Renormalisation of a hierarchical ϕ_3^4 model

1988 J. Phys. A: Math. Gen. 21 1753

(http://iopscience.iop.org/0305-4470/21/8/011)

J. Phys. A: Math. Gen. 21 (1988) 1753-1758. Printed in the UK

Renormalisation of a hierarchical ϕ_3^4 model†

T C Dorlas

School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

Received 23 November 1987

Abstract. We define a hierarchical model in $d > 2$ dimensions with a ϕ^4 interaction which is a true model of statistical mechanics in that it can be described by a set of potentials admitting Gibbs states. We show that the renormalisation of this model leads to a transformation of local potentials which corresponds to a transformation of Gibbs measures in the thermodynamic limit. Finally we show how this transformation can be used to obtain a continuum limit in the three-dimensional case, using analyticity techniques developed by Gawedzki and Kupiainen.

1. Introduction

We study a hierarchical model analogous to the model introduced by Baker [1] who first pointed out its simple renormalisation group structure. A hierarchical model with discrete spins was introduced earlier by Dyson [2]. Mathematical investigation of hierarchical models and their renormalisation was initiated by Bleher and Sinai [3] and elaborated by Collet and Eckmann [4]. More recently Gawedzki and Kupiainen [5] developed powerful techniques to treat these models using analyticity properties of the interaction. In the following we study the continuum limit of a hierarchical model with ϕ^4 interaction in three dimensions, and state the final existence result which can be obtained using these techniques. Details of the proof can be found in the author's thesis [6]. Our hierarchical model differs from the one introduced in [5] in that it can be described by a Hamiltonian that admits Gibbs states. The existence of a continuum limit for the translation-invariant ϕ_3^4 model is already long established [7, 8]. However, the renormalisation group method illustrates nicely the origin and necessity of the counterterms that have to be added to the so-called 'naked' Hamiltonian. Due to the asymptotic freedom in the ultraviolet it turns out to be possible to construct strongly interacting continuum theories.

2. The model and the renormalisation transformation

We first define a Gaussian measure γ_C on $\boldsymbol{R}^{\boldsymbol{Z}^d}$ with a hierarchical covariance C given by

$$C_{xy} = \sum_{k=0}^{\infty} L^{-2\sigma k}\Gamma(x^{(k)}, y^{(k)}). \tag{2.1}$$

† Presented at the conference on *Mathematical Problems in Statistical Mechanics* held at Heriot-Watt University on 3-5 August 1987.

0305-4470/88/081753+06$02.50

Here $L > 1$ is a fixed integer, and we have divided the lattice $\mathbf{Z}^d$ into blocks $B_L(\dot{x})$ of side L indexed by $\dot{x} \in \mathbf{Z}^d$:

$$B_L(\dot{x}) = \{x \in \mathbf{Z}^d \,|\, -\tfrac{1}{2}L < x - L\dot{x} \leqslant \tfrac{1}{2}L\} \tag{2.2}$$

where $\dot{x}$ is the index of the block containing x, i.e. $x \in B_L(\dot{x})$ and $x^{(k)} = \dot{x}^{(k-1)}$. The matrix $\Gamma(x, y)$ is defined by

$$\Gamma(x, y) = \begin{cases} 1 - L^{-d} & \text{if } x = y \\ -L^{-d} & \text{if } \dot{x} = \dot{y},\, x \neq y \\ 0 & \text{if } \dot{x} \neq \dot{y}. \end{cases} \tag{2.3}$$

Note that C is not translation invariant. One easily shows that C_{xy} behaves as $L^{-2\sigma s(x,y)}$ for $s(x, y) \to \infty$, where $s(x, y) + 1$ is the smallest $k \geqslant 1$ so that $x^{(k)} = y^{(k)}$. Thus $\delta(x, y) = L^{s(x,y)}$ is a kind of 'hierarchical distance' between x and y. Taking $\sigma = \frac{1}{2}(d-2)$, C_{xy} mimics the power-law behaviour of $(-\Delta)^{-1}(\underset{\sim}{x}, \underset{\sim}{y}) =$ constant $\times |\underset{\sim}{x} - \underset{\sim}{y}|^{-d+2}$. One can show that γ_C satisfies the DLR conditions [9, 10] with respect to the following potentials $\mathcal{V}_X$ (for finite $X \subset \mathbf{Z}^d$):

$$\begin{aligned} \mathcal{V}_x(\phi_x) &= \tfrac{1}{2}(1 - L^{-d})/(1 - L^{-d-2}) \\ \mathcal{V}_{\{x,y\}}(\phi_x, \phi_y) &= L^{-(d+2)(s(x,y)+1)}(L^2 - 1)/(1 - L^{-d-2}) \\ \mathcal{V}_X &= 0 \qquad \text{if } |X| > 2. \end{aligned} \tag{2.4}$$

Proposition 1. The Gaussian measure γ_C, with hierarchical covariance C given by (2.1) with $\sigma = \frac{1}{2}(d-2)$, is a Gibbs measure with respect to the potentials (2.4).

The main ingredient of the proof is the fact that

$$\sum_{y \in \mathbf{Z}^d} |B_{xy}| = 2(1 - L^{-d})/(1 - L^{-2}) < \infty \tag{2.5}$$

where

$$B_{xy} = \sum_{k=0}^{\infty} L^{-(d+2)k}\Gamma(x^{(k)}, y^{(k)}) \tag{2.6}$$

is the inverse of the matrix C.

Let us now add a local interation to $\mathcal{V}$. Define

$$V(\phi) = \sum_{x \in \mathbf{Z}^d} v(\phi_x) \tag{2.7}$$

and put

$$v(\phi_x) = \tfrac{1}{2}r\phi_x^2 + \tfrac{1}{4}g\phi_x^4. \tag{2.8}$$

Using Dobrushin's existence theorem [11] (see also [12]) one can prove (see [6]) the following theorem.

Theorem 1. There exists a Gibbs state μ for the hierarchical ϕ^4 model defined by the potentials

$$\begin{aligned} \mathcal{V}_x(\phi_x) &= \tfrac{1}{2}(1 - L^{-d})/(1 - L^{-d-2}) + v(\phi_x) \\ \mathcal{V}_{\{x,y\}}(\phi_x, \phi_y) &= L^{-(d+2)(s(x,y)+1)}(L^2 - 1)/(1 - L^{-d-2}) \\ \mathcal{V}_X &= 0 \qquad \text{if } |X| > 2. \end{aligned} \tag{2.9}$$

The hierarchical covariance has the nice property that, under the (block-spin) renormalisation transformation, local interactions, i.e. interactions of the form (2.7), are conserved. Indeed, let us define a block-spin transformation as follows. Given $\phi \in R^{Z^d}$ we define a block average $M\phi = \phi'$ by

$$(M\phi)_x = L^{-d+\sigma} \sum_{y \in B_L(x)} \phi_y. \tag{2.10}$$

The transformed measure $\mu' = R\mu$ is then the image measure under the mapping M: $\mu' = M(\mu)$. Note that

$$R\gamma_C = \gamma_C. \tag{2.11}$$

Indeed, one easily calculates the characteristic function

$$\begin{aligned}\int \exp(\mathrm{i}\langle\phi, f\rangle) R\gamma_C(\mathrm{d}\phi) &= \int \exp(\mathrm{i}\langle M\phi, f\rangle)\gamma_C(\mathrm{d}\phi) \\ &= \int \exp(\mathrm{i}\langle\phi, M^{\mathrm{t}} f\rangle)\gamma_C(\mathrm{d}\phi) \\ &= \exp(-\tfrac{1}{2}\langle M^{\mathrm{t}} f, CM^{\mathrm{t}} f\rangle).\end{aligned}$$

Hence $R\gamma_C$ is a Gaussian measure with covariance

$$C' = MCM^{\mathrm{t}} = C. \tag{2.12}$$

We now show that μ' is also a Gibbs measure for the hierarchical model, but with a transformed local potential $V'(\phi') = \Sigma_{x \in Z^d} v'(\phi'_x)$. Let us first make a heuristic calculation. Formally, we have, for an arbitrary function F on $\Omega_\Lambda = \boldsymbol{R}^\Lambda$ with $\Lambda \subset \boldsymbol{Z}^d$ finite,

$$\int F(\phi_\Lambda)\mu(\mathrm{d}\phi) = \frac{\int F(\phi_\Lambda) \exp[-\Sigma_{x \in Z^d} v(\phi_x)]\gamma_C(\mathrm{d}\phi)}{\int \exp[-\Sigma_{x \in Z^d} v(\phi_x)]\gamma_C(\mathrm{d}\phi)}.$$

Now

$$C_{xy} = L^{-2\sigma} C_{x'y'} + \Gamma_{xy} \tag{2.13}$$

so let us put

$$\phi_x = L^{-\sigma}\phi'_{x'} + \xi_x \tag{2.14}$$

and

$$\gamma_C(\mathrm{d}\phi) = \gamma_C(\mathrm{d}\phi')\gamma_\Gamma(\mathrm{d}\xi). \tag{2.15}$$

The field ξ is a fluctuation field satisfying $M\xi = 0$, i.e. γ_Γ is concentrated on the ξ with zero average on each block. The decomposition (2.15) was first introduced by Sinai [13] and used extensively by Gawedzki and Kupiainen [14, 15]. Using this decomposition we can compute the renormalised measure μ' as follows:

$$\begin{aligned}\int F(\phi'_\Lambda)\mu'(\mathrm{d}\phi') &= \int F((M\phi)_\Lambda)\mu(\mathrm{d}\phi) \\ &= \frac{\int F((M\phi)_\Lambda) \exp[-\Sigma_x v(\phi_x)]\gamma_C(\mathrm{d}\phi)}{\int \exp[-\Sigma_x v(\phi_x)]\gamma_C(\mathrm{d}\phi)} \\ &= \frac{\iint F(\phi'_\Lambda) \exp[-\Sigma_x v(L^{-\sigma}\phi'_{x'} + \xi_x)]\gamma_C(\mathrm{d}\phi')\gamma_\Gamma(\mathrm{d}\xi)}{\iint \exp[-\Sigma_x v(L^{-\sigma}\phi'_{x'} + \xi_x)]\gamma_C(\mathrm{d}\phi')\gamma_\Gamma(\mathrm{d}\xi)}.\end{aligned}$$

Now using the fact that γ_C decouples over the blocks so that $\gamma_\Gamma = \bigotimes_{\dot{x}} \gamma_{\Gamma_{\dot{x}}}$, we find

$$\int F(\phi'_\Lambda)\mu'(\mathrm{d}\phi')$$

$$= \frac{\int \gamma_C(\mathrm{d}\phi')F(\mathrm{d}\phi'_\Lambda)\Pi_{\dot{x}} \int \gamma_{\Gamma_{\dot{x}}}(\mathrm{d}\xi) \exp[-\Sigma_{x\in B_L(\dot{x})} v(L^{-\sigma}\phi'_{\dot{x}} + \xi_x)]}{\int \gamma_C(\mathrm{d}\phi')\Pi_{\dot{x}} \int \gamma_{\Gamma_{\dot{x}}}(\mathrm{d}\xi) \exp[-\Sigma_{x\in B_L(\dot{x})} v(L^{-\sigma}\phi'_{\dot{x}} + \xi_x)]}.$$

Hence we expect that μ' is a Gibbs measure for the hierarchical model with local potential v' given by

$$\exp(-v'(\phi'_{\dot{x}})) = \frac{\int \exp[-\Sigma_{x\in B_L(\dot{x})} v(L^{-\sigma}\phi'_{\dot{x}} + \xi_x)]\gamma_{\Gamma_{\dot{x}}}(\mathrm{d}\xi)}{\int \exp[-\Sigma_{x\in B_L(\dot{x})} v(\xi_x)]\gamma_{\Gamma_{\dot{x}}}(\mathrm{d}\xi)}. \tag{2.16}$$

We have normalised v' so that $v'(0)=0$. We remark that $\Gamma_{\dot{x}} = \Gamma_0$ is independent of $\dot{x}$; it is the restriction of Γ to a block. Therefore v' does not depend on $\dot{x}$ and we can omit all indices $\dot{x}$ in (2.16). It is now easy to prove the following rigorous statement.

Proposition 2. Assume that μ is a Gibbs measure for the hierarchical model with potentials (2.9). Then $\mu' = R\mu$ is a Gibbs measure for the hierarchical model with potentials (2.9) modified by replacing v with v' given by (2.16).

Proof. If μ is a Gibbs measure for the hierarchical model with local potential v then

$$\mu(\mathrm{d}\phi_\Lambda|\phi_{\Lambda^c}) = \frac{\exp[-\Sigma_{x\in\Lambda} v(\phi_x)]\gamma(\mathrm{d}\phi_\Lambda|\phi_{\Lambda^c})}{\int \exp[-\Sigma_{x\in\Lambda} v(\phi_x)]\gamma(\mathrm{d}\phi_\Lambda|\phi_{\Lambda^c})} \tag{2.17}$$

where $\gamma(\mathrm{d}\phi_\Lambda|\phi_{\Lambda^c})$ denotes the conditional distribution of ϕ_Λ given ϕ_{Λ^c}. We want to prove

$$\mu'(\mathrm{d}\phi'_\Lambda|\phi'_{\Lambda^c}) = \frac{\exp[-\Sigma_{\dot{x}\in\Lambda} v'(\phi'_{\dot{x}})]\gamma(\mathrm{d}\phi'_\Lambda|\phi'_{\Lambda^c})}{\int \exp[-\Sigma_{\dot{x}\in\Lambda} v'(\phi'_{\dot{x}})]\gamma(\mathrm{d}\phi'_\Lambda|\phi'_{\Lambda^c})} \tag{2.18}$$

or, equivalently,

$$\int_J \mu'(\mathrm{d}\phi'_{\Lambda^c}) \frac{\int_I \exp[-\Sigma_{\dot{x}\in\Lambda} v'(\phi'_{\dot{x}})]\gamma(\mathrm{d}\phi'_\Lambda|\phi'_{\Lambda^c})}{\int \exp[-\Sigma_{\dot{x}\in\Lambda} v'(\phi'_{\dot{x}})]\gamma(\mathrm{d}\phi'_\Lambda|\phi'_{\Lambda^c})} = \int_{I\times J} \mu'(\mathrm{d}\phi)$$

for all measurable $I \subset \Omega_\Lambda$, $J \subset \Omega_{\Lambda^c}$.

Now, because of the independence of ξ and ϕ', we have

$$\int\int F((L^{-\sigma}\phi'_{\dot{x}} + \xi_x)_{x\in\bar{\Lambda}}) \prod_{\dot{x}\in\Lambda} \gamma_{\Gamma_{\dot{x}}}(\mathrm{d}\xi)\gamma(\mathrm{d}\phi'_\Lambda|\phi'_{\Lambda^c})$$

$$= \int \gamma_\Gamma(\mathrm{d}\xi_{\bar{\Lambda}^c}) \int F(\phi_{\bar{\Lambda}})\gamma(\mathrm{d}\phi_{\bar{\Lambda}}|(L^{-\sigma}\phi'_{\dot{y}} + \xi_y)_{y\in\bar{\Lambda}^c}). \tag{2.19}$$

Here $\bar{\Lambda}$ is the union of blocks labelled by the points of Λ, i.e. $\bar{\Lambda} = \bigcup_{x\in\Lambda} B_L(x)$. Using (2.19) and the transformation formula (2.16) it follows that

$$\frac{\int_I \exp[-\Sigma_{\dot{x}\in\Lambda} v(\phi'_{\dot{x}})]\gamma(\mathrm{d}\phi'_\Lambda|\phi'_{\Lambda^c})}{\int \exp[-\Sigma_{\dot{x}\in\Lambda} v(\phi'_{\dot{x}})]\gamma(\mathrm{d}\phi'_\Lambda|\phi'_{\Lambda^c})}$$

$$= \frac{\int \gamma_\Gamma(\mathrm{d}\xi_{\bar{\Lambda}^c}) \int_{M^{-1}(I)} \exp[-\Sigma_{x\in\bar{\Lambda}} v(\phi_x)]\gamma(\mathrm{d}\phi_{\bar{\Lambda}}|(L^{-\sigma}\phi'_{\dot{y}} + \xi_y)_{y\in\bar{\Lambda}^c})}{\int \gamma_\Gamma(\mathrm{d}\xi_{\bar{\Lambda}^c}) \int \exp[-\Sigma_{x\in\bar{\Lambda}} v(\phi_x)]\gamma(\mathrm{d}\phi_{\bar{\Lambda}}|(L^{-\sigma}\phi'_{\dot{y}} + \xi_y)_{y\in\bar{\Lambda}^c})}. \tag{2.20}$$

Now B_{xy} does not depend on $y - L\dot{y}$ if $x \neq y$ (see (2.6)). It follows that the measure $\gamma(\mathrm{d}\phi_{\bar{\Lambda}}|(L^{-\sigma}\phi'_{\dot{y}} + \xi_y)_{y\in\bar{\Lambda}^c})$ is in fact independent of ξ. Integrating (2.20) with respect to ϕ'_{Λ^c} and using (2.17), the validity of (2.18) easily follows.

3. The continuum limit

It is well known that the renormalisation group provides a means of constructing a continuum limit. This can be done in a two-step process. First one constructs a sequence of lattice fields ϕ_n satisfying the consistency condition

$$(\phi_n)_x = L^{-d} \sum_{y \in B_L(x)} (\phi_{n+1})_y. \tag{3.1}$$

These can be rescaled to obtain fields $\varphi_n(\underline{x}) = (\phi_n)_{L^n \underline{x}}$ with $\underline{x} \in L^{-n}\boldsymbol{Z}^d$ on a sequence of finer and finer lattices. The second step consists of proving the existence of the limit $\lim_{n\to\infty} \varphi_n(f) = \varphi(f)$ for a class of smooth functions f, e.g., $f \in \mathcal{S}(\boldsymbol{R}^d)$. (For an elementary discussion of this process, see [16].) We concentrate here on the first step in the process; this involves the renormalisation of the field.

The lattice fields ϕ_n can be obtained in the following way: let $\phi_{(m)}$ be a lattice approximation to the continuum field we wish to construct, with variable parameters depending on m; we obtain ϕ_n as the limit

$$\phi_n = L^{n\sigma} \lim_{m\to\infty} M^{m-n} \phi_{(m)}. \tag{3.2}$$

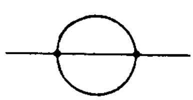
The distribution of the field $\phi_{(m)}$ will be given by the potential (2.8) but with parameters r_m and g_m depending on m. In order to see how r_m and g_m have to be varied with m, we apply the renormalisation transformation (2.16) once. To second order in perturbation theory we obtain

$$\begin{aligned} r' &= L^2(r + 3ag - 3arg - 9a^2g^2 - 6cg^2) \\ g' &= L^{4-d}(g - 9ag^2) \end{aligned} \tag{3.3}$$

where a and c are constants: $a = 1 - L^{-d}$; $c = (1 - L^{-d})(1 - 2L^{-d})$. Let us consider the case $d = 3$. We can read off from the transformation formulae (3.3) the way in which r_m and g_m have to depend on m. Indeed, taking

$$\begin{aligned} r_m &= L^{-2m}(r_0 - 3a\gamma_m L^m g_0 + 6cmg_0^2) \\ g_m &= L^{-m} g_0 \end{aligned} \tag{3.4}$$

with $\gamma_m = (1 - L^{-m})/(1 - L^{-1})$, we find that $(r_m^{(m-n)}, g_m^{(m-n)})$ converges as $m \to \infty$. In fact we can already guess from the the form of the equations (3.3) that (3.4) suffices to all orders of perturbation theory since higher-order terms in (3.3) are all of order $\leq L^{-m}$ if we insert (3.4). This is also true for ϕ'^6 terms, ϕ'^8 terms, etc, that appear in $v'(\phi')$. Notice that the first counterterm in the expression for r_m is just a Wick-reordering term, i.e. it results from replacing ϕ^4 by $:\phi^4:$ in (2.8). The second counterterm is the usual mass-renormalisation term corresponding to the logarithmically diverging Feynman diagram



We remark here that, not only is (3.4) correct to all orders of perturbation theory, it is even correct non-perturbatively. This can be proven using analyticity techniques developed by Gawedzki and Kupiainen [5]. To make this statement precise, let us define the initial local potentials

$$v_m(\varphi) = \tfrac{1}{2}r_m\varphi^2 + \tfrac{1}{4}g_m\varphi^4 \tag{3.5}$$

where r_m and g_m are given by (3.4) with $g_0 > 0$. Then the following holds.

Theorem 2. Assume L large enough. Then there exists $m_0(L, r_0, g_0)$ such that, if $m \geqslant m_0$, $\exp[-(\mathscr{R}^n v_{m+n})(\varphi)]$ converges uniformly in $\varphi \in \boldsymbol{R}$ as $n \to \infty$, where $\mathscr{R}$ is the transformation (2.16).

For a proof, see [6]. Notice that there is no restriction on r_0 and $g_0 > 0$. This is because, for large enough m, r_m and g_m will be small, so that we can do perturbation theory in r_m and g_m. The fact that r_m and g_m approach zero as $m \to \infty$ is the ultraviolet asymptotic freedom of the theory.

Acknowledgments

The author thanks Professor N M Hugenholtz and Professor J T Lewis for helpful remarks. He also thanks Professor O Penrose for inviting him to speak at a most enjoyable conference. Part of this work was carried out under the auspices of the Stichting voor Fundamenteel Onderzoek der Materie (FOM), which is financially supported by the Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (ZWO).

References

[1] Baker G A 1972 *Phys. Rev.* B **5** 2622
[2] Dyson F J 1969 *Commun. Math. Phys.* **12** 91
[3] Bleher P M and Sinai Ya G 1973 *Commun. Math. Phys.* **33** 23
[4] Collet P and Eckmann J P 1978 *A Renormalization Group Analysis of the Hierarchical Model in Statistical Mechanics* (*Lecture Notes in Mathematics* **74**) (Berlin: Springer)
[5] Gawedzki K and Kupiainen A 1982 *J. Stat. Phys.* **29** 683
[6] Dorlas T C 1987 *Thesis* University of Groningen
[7] Glimm J and Jaffe A 1973 *Fortschr. Phys.* **21** 327
[8] Feldman J and Osterwalder K 1976 *Ann. Phys., NY* **97** 80
[9] Dobrushin R L 1968 *Theor. Prob. Appl.* **13** 197
[10] Dobrushin R L 1970 *Theor. Prob. Appl.* **15** 458
[11] Lanford O E and Ruelle D 1969 *Commun. Math. Phys.* **13** 194
[12] Sinai Ya G 1982 *Theory of Phase Transitions: Rigorous Results* (Oxford: Pergamon)
[13] Sinai Ya G 1976 *Theor. Prob. Appl.* **21** 64
[14] Gawedzki K and Kupiainen A 1980 *Commun. Math. Phys.* **77** 31
[15] Gawedzki K and Kupiainen A 1983 *Ann. Phys., NY* **47** 198
[16] Dorlas T C 1988 *Proc. of Mark Kac Seminar* ed W T F den Hollander and H Maassen (Amsterdam: Mathematisch Centrum)